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On \mathcal{H}_∞ Model Reduction for Discrete-Time LTI Systems Using LMIs

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Abstract

In this paper, we address the \mathcal{H}_∞ model reduction problem for linear time-invariant discrete-time systems. We revisit this problem by means of linear matrix inequality (LMI) approaches and first show a concise proof for the well-known lower bounds on the approximation error, which is given in terms of the Hankel singular values of the system to be reduced. In addition, when we reduce the system order by the multiplicity of the smallest Hankel singular value, we show that the \mathcal{H}_∞ optimal reduced-order model can readily be constructed via LMI optimization. These results can be regarded as complete counterparts of those recently obtained in the continuous-time system setting.

Keywords: \mathcal{H}_∞ model reduction, discrete-time LTI systems, LMIs.

1 Introduction

In this paper, we address the \mathcal{H}_∞ model reduction problem for linear-time invariant (LTI) discrete-time systems. For given system $G(z) \in \mathcal{RH}_\infty$ of McMillan degree n , the problem is to find a system $G_r(z) \in \mathcal{RH}_\infty$ of McMillan degree $r(< n)$ that minimizes $\|G - G_r\|_\infty$. As is well-known, model reduction problems for both the continuous- and discrete-time LTI systems has been a central topic in control theory and effective methodologies such as Hankel norm approximation method [10, 11, 17, 26, 27] and balanced truncation method [1, 11, 26, 27] have been proposed. However, constructing \mathcal{H}_∞ optimal reduced-order models in analytic form should be far from attainable in general cases and still remains open to this date. This arises recent intensive studies on model reduction via numerical optimization, in particular by means of linear matrix inequality (LMI) optimization.

In most of the existing approaches by means of LMIs, the \mathcal{H}_∞ model reduction problems are first recast into optimization problems subject to bilinear matrix inequalities (BMIs) via KYP lemma [2, 22]. The resulting optimization problems are highly non-convex but have similar structural properties to those arising in reduced-order (or fixed-order) controller synthesis problems. Thus, those iterative algorithms [9, 12, 13, 15] developed for reduced-order controller synthesis can be applied to find suboptimal models. These LMI-based approaches

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are also extended to frequency weighted \mathcal{H}_∞ model reduction [23], model reduction of singular systems [24], Markovian jump linear systems [25], and discrete time-delay systems [6]. On the other hand, the LMI methods recently proposed by Geromel et al. [7, 8] are tailored to solve model reduction problems for LTI systems and do not require to solve LMIs iteratively. These methods would be promising to obtain good suboptimal models with reasonable computational effort, although there is no rigorous results on how far the resulting models are close to the optimal one.

In the present paper, we address the discrete-time \mathcal{H}_∞ model reduction problems using LMIs as well. However, the goal is not to pursue heuristic numerical optimization procedures but to obtain “analytic” results by means of LMI-related techniques. In particular, we first obtain lower bounds of the error incurred in approximating $G(z)$ by $G_r(z)$, i.e., the lower bounds for $\|G - G_r\|_\infty$ by combining the LMI results for model reduction [12] and the well-known Lyapunov equalities for the balanced controllability and observability Gramian. These lower bounds are given in terms of the Hankel singular values of the system $G(z)$ and exactly the same as those already known in the literature [10, 11, 17, 26, 27]. Thus we provide a concise and closed-form proof for those well-known results using LMI-related techniques. In addition, when we reduce the order by the multiplicity of the smallest Hankel singular value, we show that the \mathcal{H}_∞ optimal reduced-order models can be easily constructed via LMI optimization. It turns out that these results are complete counterparts of those recently obtained in the continuous-time system setting [3].

It should be noted that, once we have those results in [3], the lower bounds and the optimal reduced-order models in the discrete-time setting follow immediately by applying a bilinear transformation between the imaginary axis and the unit circle [19]. It is nonetheless interesting to develop LMI techniques for purely discrete-time systems and in particular, it should be significant to find out specific LMI conditions for the optimal \mathcal{H}_∞ model reduction in the aforementioned cases. Another motivation of the present work is the recent intensive studies on model reduction of linear time-varying discrete-time systems [5, 18, 21]. In view of the fact that inequality conditions on the associated linear operators plays an essential role in these studies, we expect that the results in the present paper can become effective even for model reduction of LTV systems.

We use the following notations in this paper. I_n and $0_{n,m}$ denote respectively the identity matrix of dimension n and the zero matrix of dimension $n \times m$; we omit the dimensions when no confusions occur. For a matrix $A \in \mathbf{R}^{n \times n}$, A^{-1} and A^T are the inverse and transpose of the matrix A , respectively. $\text{He}\{A\}$ is a shorthand notation for $A + A^T$. For a matrix $A \in \mathbf{R}^{n \times m}$ with $\text{rank}(A) = r < n$, $A^\perp \in \mathbf{R}^{(n-r) \times n}$ is a matrix such that $A^\perp A = 0$ and $A^\perp (A^\perp)^T > 0$. For a symmetric matrix A , we denote by triplet $(\text{In}_-(A), \text{In}_0(A), \text{In}_+(A))$ the numbers of its strictly negative, zero, and strictly positive eigenvalues, respectively. Furthermore, we denote the set of $n \times n$ positive-definite matrices by \mathcal{S}_n .

2 Balanced Realization and BMI Condition for the \mathcal{H}_∞ Model Reduction

Let us consider the \mathcal{H}_∞ model reduction problems of a discrete-time system $G(z) \in \mathcal{RH}_\infty$ whose McMillan degree is n and minimal realization is given by

$$G(z) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right], \quad A \in \mathbf{R}^{n \times n}, \quad B \in \mathbf{R}^{n \times p}, \quad C \in \mathbf{R}^{q \times n}, \quad D \in \mathbf{R}^{q \times p}. \quad (1)$$

More precisely, the \mathcal{H}_∞ model reduction problem considered in this paper is to find a system $G_r(z) \in \mathcal{RH}_\infty$ of McMillan degree r that minimizes $\|G - G_r\|_\infty$ where $r < n$. In the following, we assume that the realization in (1) is already balanced, i.e., its controllability and observability Gramians are equal and diagonal [1]. Hence, by denoting the balanced Gramian by Σ , the state space matrices A , B and C in (1) satisfy

$$-\Sigma + A\Sigma A^T + BB^T = 0, \quad (2a)$$

$$-\Sigma + A^T \Sigma A + C^T C = 0 \quad (2b)$$

where

$$\Sigma = \text{diag}(\sigma_1 I_{k_1}, \dots, \sigma_l I_{k_l}, \sigma_{l+1} I_{k_{l+1}}, \dots, \sigma_m I_{k_m}), \quad \sigma_1 > \dots > \sigma_l > \sigma_{l+1} > \dots > \sigma_m > 0. \quad (3)$$

Note that k_i is the multiplicity of σ_i and $k_1 + \dots + k_m = n$. As is well-known, the diagonal entries of Σ are called the Hankel singular values of the system $G(z)$ and plays a key role in the balanced truncation method (see, e.g., [1]).

We tackle the model reduction problems based on the state-space formulas developed for the \mathcal{H}_∞ controller synthesis [2, 22]. To this end, let us suppose that the state space matrices of the system $G_r(z)$ are given by (A_r, B_r, C_r, D_r) and write the state space realization of the error system $E(z) := G(z) - G_r(z)$ as follows:

$$E(z) = \left[\begin{array}{c|c} A_e & B_e \\ \hline C_e & D_e \end{array} \right] = \left[\begin{array}{c|c} \mathcal{A} & \mathcal{B}_1 \\ \hline \mathcal{C}_1 & \mathcal{D}_{11} \end{array} \right] + \left[\begin{array}{c|c} \mathcal{B}_2 \\ \hline \mathcal{D}_{12} \end{array} \right] \mathcal{G} \left[\begin{array}{c|c} \mathcal{C}_2 & \mathcal{D}_{21} \end{array} \right]. \quad (4)$$

Here, the matrices in (4) are given by

$$\left[\begin{array}{c|c|c} \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 \\ \hline \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{12} \\ \hline \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22} \end{array} \right] = \left[\begin{array}{cc|cc|cc} A & 0 & B & 0 & 0 & 0 \\ 0 & 0 & 0 & I_r & 0 & 0 \\ \hline C & 0 & D & 0 & 0 & -I_q \\ 0 & I_r & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I_p & 0 & 0 & 0 \end{array} \right], \quad \mathcal{G} = \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix}. \quad (5)$$

Under these notations, we now review the matrix inequality condition for the \mathcal{H}_∞ model reduction. From KYP Lemma [2, 22], we see that $\|E\|_\infty < \gamma$ holds if and only if there exists $P \in \mathbf{R}^{(n+r) \times (n+r)}$ such that

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \in \mathcal{S}_{n+r}, \quad (6)$$

$$\begin{bmatrix} A_e & B_e \\ C_e & D_e \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & I_q \end{bmatrix} \begin{bmatrix} A_e & B_e \\ C_e & D_e \end{bmatrix} < \begin{bmatrix} P & 0 \\ 0 & \gamma^2 I_p \end{bmatrix}$$

where $P_{11} \in \mathcal{S}_n$ and $P_{22} \in \mathcal{S}_r$. As we see, the above inequalities are BMIs with respect to the matrix variables P and A_r, B_r, C_r, D_r since bilinear terms occur among them. Indeed, the \mathcal{H}_∞ model reduction problems are likely to be essentially non-convex problems and thus computing globally optimal solutions remains open to this date [12].

As such, it is hard to obtain definite results for the \mathcal{H}_∞ model reduction problems if we directly work on the BMI condition (6). However, by eliminating the variable \mathcal{G} from (6) through the Parrott's Lemma [4, 20], we can obtain a more concise matrix inequality condition [12], which plays an important role in the subsequent discussions. This inequality condition is given formally in the following lemma.

Lemma 1 [12] *Let us consider a system $G(z) \in \mathcal{RH}_\infty$ of McMillan degree n and its minimal realization (1). Then, there exists $G_r(z) \in \mathcal{RH}_\infty$ of McMillan degree at most r that satisfies $\|G - G_r\|_\infty < \gamma$ if and only if there exist $X_{11} \in \mathcal{S}_n$, $P_{11} \in \mathcal{S}_n$, $P_{12} \in \mathbf{R}^{n \times r}$ and $P_{22} \in \mathcal{S}_r$ that satisfy the following matrix inequalities:*

$$-X_{11} + AX_{11}A^T + \frac{1}{\gamma^2}BB^T < 0, \quad (7a)$$

$$-P_{11} + A^T P_{11} A + C^T C < 0, \quad (7b)$$

$$X_{11} = (P_{11} - P_{12} P_{22}^{-1} P_{12}^T)^{-1}. \quad (7c)$$

It should be noted that the condition (7) is still non-convex with respect to the decision variables X_{11}, P_{11}, P_{12} and P_{22} due to the equality constraint (7c). This equality constraint commonly arises in reduced-order (fixed-order) \mathcal{H}_∞ controller synthesis and by extracting this particular structure, several effective local search algorithms have been developed [9, 12, 13, 15]. It is nonetheless hard to solve (7) exactly and thus the rearrangement from (6) into (7) does not allow us to avoid completely the essential difficulties stemming from the nonconvexity of the \mathcal{H}_∞ model reduction problem.

However, the condition (7) is surely effective to analyze lower bounds of $\|G - G_r\|_\infty$ and derive an LMI for constructing optimal \mathcal{H}_∞ models of particular reduced-order. The key observation is that the first two inequalities in (7) is closely related to the Lyapunov equalities (2) for the balanced controllability and observability Gramians. By noting this fact, we can obtain those definite results as explicated in the next section.

3 Main Results

3.1 Analysis of Lower Bounds Using LMI Techniques

Now we are in a position to state the main results of the paper. The first result concerns lower bounds for $\|G - G_r\|_\infty$. For the ease of our statements, in the following theorem, we neglect the multiplicity of the Hankel singular values of $G(z)$ given in (3) and denote them by $\sigma_1 \geq \dots \geq \sigma_r \geq \sigma_{r+1} \geq \dots \geq \sigma_n > 0$.

Theorem 1 *Let us consider a system $G(z) \in \mathcal{RH}_\infty$ of McMillan degree n with the Hankel singular values $\sigma_1 \geq \dots \geq \sigma_r \geq \sigma_{r+1} \geq \dots \geq \sigma_n > 0$. Then, for all $G_r(z) \in \mathcal{RH}_\infty$ of McMillan degree less than or equal to r , we have*

$$\|G - G_r\|_\infty \geq \sigma_{r+1}. \quad (8)$$

The following lemma is used to prove Theorem 1.

Lemma 2 [16] *For given two symmetric matrices $W \in \mathbf{R}^{n \times n}$ and $Z \in \mathbf{R}^{n \times n}$, $W < Z$ holds only if $\lambda_i(W) < \lambda_i(Z)$ ($i = 1, \dots, n$) where $\lambda_i(W)$ denotes the i -th largest eigenvalue of W .*

Proof of Theorem 1. To prove the assertion, it suffices to show that the condition (7) does not hold if $\gamma \leq \sigma_{r+1}$. From (2) with the lefthand side of (2a) divided by γ^2 and the first two inequalities in (7), we readily obtain

$$-(X_{11} - \frac{1}{\gamma^2}\Sigma) + A(X_{11} - \frac{1}{\gamma^2}\Sigma)A^T < 0, \quad -(P_{11} - \Sigma) + A^T(P_{11} - \Sigma)A < 0. \quad (9)$$

Since the matrix A is Schur stable, it follows that

$$X_{11} - \frac{1}{\gamma^2}\Sigma > 0, \quad P_{11} - \Sigma > 0. \quad (10)$$

The rest of the proof is exactly the same as those in [3]. Indeed, from (10) and (7c), we see that the following condition is necessary for the condition (7) to hold:

$$\Sigma - \gamma^2 \Sigma^{-1} < P_{12} P_{22}^{-1} P_{12}^T. \quad (11)$$

If $\gamma \leq \sigma_{r+1}$, however, we see from the diagonal entries of $\Sigma - \gamma^2 \Sigma^{-1}$ that $\text{In}_-(\Sigma - \gamma^2 \Sigma^{-1}) \leq n - r - 1$ whereas it is apparent that $\text{In}_0(P_{12} P_{22}^{-1} P_{12}^T) \geq n - r$. Thus, from Lemma 2, the condition (11) cannot be satisfied if $\gamma \leq \sigma_{r+1}$. This completes the proof. Q.E.D.

It should be noted that the lower bound given in Theorem 1 is already known in the optimal Hankel norm approximation method [10, 17]. In stark contrast with these existing studies where the lower bounds were derived for the approximation error measured by the Hankel norm, we arrived at the results (8) by directly considering the \mathcal{H}_∞ norm of the error system. The proof here should be fairly concise, and we have observed that recently developed LMI-related techniques work effectively in conjunction with the basic results from linear algebra.

3.2 Optimal \mathcal{H}_∞ Model Reduction via LMI Optimization

In the preceding subsection, we have proved that $\|G - G_r\| \geq \sigma_{r+1}$ holds for all $G_r(z) \in \mathcal{RH}_\infty$ of McMillan degree less than or equal to r . To strengthen this result, it would be desirable to show the exactness of the lower bounds. The goal of this subsection is to show that, in the case where we construct reduced-order models of order $r = n - k_m$, i.e, if we reduce the system order by the multiplicity of the smallest Hankel singular value, this lower bound is indeed the infimum. This result is already known in the optimal Hankel norm approximation method [10, 11, 17] but our proof here is entirely different from those existing ones. It follows that this particular proof enables us to see that the optimal reduced-order model that attains this infimum can be constructed via LMI optimization.

To begin with, let us follow the discussions in [3] and again focus on the Lyapunov equalities in (2). Then, it is a direct consequence that the pair $(\frac{1}{\sigma_m^2}\Sigma, \Sigma)$ satisfies the following equalities:

$$-\frac{1}{\sigma_m^2}\Sigma + A\frac{1}{\sigma_m^2}\Sigma A^T + \frac{1}{\sigma_m^2}BB^T = 0, \quad (12a)$$

$$-\Sigma + A^T\Sigma A + C^TC = 0. \quad (12b)$$

Furthermore, with respect to the equality condition (7c), the pair $(\frac{1}{\sigma_m^2}\Sigma, \Sigma)$ satisfies

$$\frac{1}{\sigma_m^2}\Sigma = (\Sigma - P_{12}P_{22}^{-1}P_{12}^T)^{-1} \quad (13)$$

with

$$P_{12} = \begin{bmatrix} I_{n-k_m} \\ 0_{k_m, n-k_m} \end{bmatrix}, \quad P_{22} = \text{diag} \left(\left(\sigma_1 - \frac{\sigma_m^2}{\sigma_1} \right)^{-1} I_{k_1}, \dots, \left(\sigma_{m-1} - \frac{\sigma_m^2}{\sigma_{m-1}} \right)^{-1} I_{k_{m-1}} \right) > 0, \quad (14)$$

which has already observed in the continuous-time setting [3]. The implication of (12) and (13) is that, in the case where $r = n - k_m$, the conditions in (7) will be satisfied for $\gamma = \sigma_m$ with $X_{11} = \frac{1}{\sigma_m^2}\Sigma$, $P_{11} = \Sigma$ and P_{12} and P_{22} given in (14), provided that we replace the inequalities in (7) to equalities. At this stage, we cannot conclude that σ_m is the infimum of $\|G - G_{n-k_m}\|_\infty$ but the above discussions can be made more rigorous and we can obtain the following results.

Lemma 3 *Let us consider a system $G(z) \in \mathcal{RH}_\infty$ of McMillan degree n with the Hankel singular values given in (3). Then, for arbitrary $\gamma > \sigma_m$, there exists $G_{n-k_m}(z) \in \mathcal{RH}_\infty$ of McMillan degree at most $n - k_m$ that satisfies $\|G - G_{n-k_m}\|_\infty < \gamma$.*

Proof. See the appendix section for the proof.

Q.E.D.

From Theorem 1 and Lemma 3, we can conclude that σ_m is indeed the infimum of $\|G - G_{n-k_m}\|_\infty$. The proof of Lemma 3 given in the appendix section has some similarities to the corresponding proof for the continuous-time systems [3], even though the proof here is rather

involved. The difficulty lies in the fact that the structure of the matrix inequality (7a) does not allow us to rewrite it into a matrix inequality with respect to $X_{11}^{-1} = P_{11} - P_{12}P_{22}^{-1}P_{12}^T$ by simply applying the congruence transformation with X_{11}^{-1} (this is surely possible in the continuous-time system setting). Thus we need another effort, and we have avoided the difficulty by applying Schur complement arguments in a particular way. On the other hand, similarly to what we have observed in the continuous-time system cases [3], it is an interesting fact that a matrix that satisfies an algebraic Riccati equation again plays a central role in the proof. It is also true that the proof heavily relies on the equalities (12) and (13). These equalities are obtained particularly for $r = n - k_m$, and unfortunately, similar equalities are not easily available in other cases. Thus, we cannot say anything on the strictness of the lower bounds given in Theorem 1 when $r < n - k_m$.

In the rest of section, we show that the optimal reduced-order model G_{n-k_m} that achieves the infimal approximation error can be constructed via LMI optimization. One important implication of the proof of Lemma 3 is that, in the case where $r = n - k_m$, we can fix the matrix variable P_{12} in (7) to be constant as in (14) without introducing any conservatism. If P_{12} is fixed, however, we can recast the non-convex matrix inequalities (7) derived in [12] into LMIs via Schur complement arguments. Once the matrix variables (P_{11}, P_{12}, P_{22}) that satisfy (7) can be found, the optimal reduced-order models can be reconstructed by solving (6) for unknown (A_r, B_r, C_r, D_r) . In this way, the \mathcal{H}_∞ optimal reduced-order models can be obtained by solving LMI optimization/feasibility problems.

Theorem 2 *The reduced-order model $G_{n-k_m}(z)$ of McMillan degree at most $n - k_m$ that minimizes $\|G - G_{n-k_m}\|_\infty$ can be obtained by the two-step procedure:*

1. Minimize γ^2 subject to the LMIs:

$$\begin{aligned} & \begin{bmatrix} P_{11} & P_{12}Q_{22} \\ Q_{22}P_{12}^T & Q_{22} \end{bmatrix} > 0, \\ & \begin{bmatrix} -(P_{11} - P_{12}Q_{22}P_{12}^T) & (P_{11} - P_{12}Q_{22}P_{12}^T)A & (P_{11} - P_{12}Q_{22}P_{12}^T)B \\ * & -(P_{11} - P_{12}Q_{22}P_{12}^T) & 0 \\ * & * & -\gamma^2 I \end{bmatrix} < 0, \\ & -P_{11} + A^T P_{11} A + C^T C < 0 \end{aligned} \quad (15)$$

where $P_{11} \in \mathcal{S}_n$ and $Q_{22} \in \mathcal{S}_{n-k_m}$ are matrix variables to be determined whereas $P_{12} \in \mathbf{R}^{n \times (n-k_m)}$ is a constant matrix given by $P_{12} = \begin{bmatrix} I_{n-k_m} \\ 0_{k_m, n-k_m} \end{bmatrix}$. For the subsequent step,

define $\tilde{P} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & Q_{22}^{-1} \end{bmatrix}$ and denote the optimal value of γ by γ_{opt} .

2. Obtain (A_r, B_r, C_r, D_r) by solving (6), where P is fixed to \tilde{P} and γ to γ_{opt} .

The LMIs in (15) given in the first step can be obtained from (7) by defining $Q_{22} := P_{22}^{-1}$ and applying Schur complements arguments. The coefficient matrices (A_r, B_r, C_r, D_r) in the second step can be constructed also from \tilde{P} by analytic formulas given in [12, 14, 22]. It should be noted that, since the choice of P_{12} depends on the state-space realizations, the result in Theorem 2 is valid only if (A, B, C) is balanced.

In our preceding results for continuous-time systems [3], it has been shown that the corresponding \mathcal{H}_∞ optimal reduced-order model can be constructed via one-step LMI optimization procedure. This is surely possible also in the discrete-time setting, and the rest of this section is devoted to the technical details to derive the desired LMIs.

From the Schur complement arguments, we can rewrite (6) equivalently as follows:

$$\begin{bmatrix} -P_{11} & -P_{12} & 0 & A^T P_{11} & A^T P_{12} & C^T \\ * & -P_{22} & 0 & A_r^T P_{12}^T & A_r^T P_{22} & -C_r^T \\ * & * & -\gamma^2 I_p & B^T P_{11} + B_r^T P_{12}^T & B^T P_{12} + B_r^T P_{22} & D^T - D_r^T \\ * & * & * & -P_{11} & -P_{12} & 0 \\ * & * & * & * & -P_{22} & 0 \\ * & * & * & * & * & -I_q \end{bmatrix} < 0. \quad (16)$$

By the similarity transformation $\bar{A}_r := P_{22} A_r P_{22}^{-1}$, $\bar{B}_r := P_{22} B_r$ and $\bar{C}_r := C_r P_{22}^{-1}$, we see that there exist (A_r, B_r, C_r, D_r) that satisfy (16) if and only if

$$\begin{bmatrix} -P_{11} & -P_{12} & 0 & A^T P_{11} & A^T P_{12} & C^T \\ * & -P_{22} & 0 & P_{22} \bar{A}_r^T P_{22}^{-1} P_{12}^T & P_{22} \bar{A}_r^T & -P_{22} \bar{C}_r^T \\ * & * & -\gamma^2 I_p & B^T P_{11} + \bar{B}_r^T P_{22}^{-1} P_{12}^T & B^T P_{12} + \bar{B}_r^T & D^T - D_r^T \\ * & * & * & -P_{11} & -P_{12} & 0 \\ * & * & * & * & -P_{22} & 0 \\ * & * & * & * & * & -I_q \end{bmatrix} < 0. \quad (17)$$

By the congruence transformation with $\text{diag}(I, Q_{22}, I, I, Q_{22}, I)$ where $Q_{22} := P_{22}^{-1}$, the above inequality reduces to

$$\begin{bmatrix} -P_{11} & -P_{12} Q_{22} & 0 & A^T P_{11} & A^T P_{12} Q_{22} & C^T \\ * & -Q_{22} & 0 & \bar{A}_r^T Q_{22} P_{12}^T & \bar{A}_r^T Q_{22} & -\bar{C}_r^T \\ * & * & -\gamma^2 I_p & B^T P_{11} + \bar{B}_r^T Q_{22} P_{12}^T & B^T P_{12} Q_{22} + \bar{B}_r^T Q_{22} & D^T - D_r^T \\ * & * & * & -P_{11} & -P_{12} Q_{22} & 0 \\ * & * & * & * & -Q_{22} & 0 \\ * & * & * & * & * & -I_q \end{bmatrix} < 0. \quad (18)$$

Here, since we can fix the matrix variable P_{12} to be constant as in (14), we see that the above inequality is nothing but an LMI with respect to the matrix variables P_{11} , Q_{22} and $\tilde{A}_r := Q_{22} \bar{A}_r$, $\tilde{B}_r := Q_{22} \bar{B}_r$, \tilde{C}_r , D_r . Once these variables have been found, the optimal reduced-order models can be reconstructed by

$$G_r(z) = \left[\begin{array}{c|c} \frac{Q_{22}^{-1} \tilde{A}_r}{\tilde{C}_r} & \frac{Q_{22}^{-1} \tilde{B}_r}{D_r} \end{array} \right]. \quad (19)$$

In the above discussions, the core to derive LMIs (15) and (18) is the fact that we can fix the matrix variable P_{12} to be constant without introducing any conservatism. It is interesting that exactly the same matrix P_{12} can be used for the construction of the optimal \mathcal{H}_∞ models as in the continuous-time system setting [3].

4 Numerical Example

Let us consider the fourth-order discrete-time system $G(z)$ with the following state-space matrices:

$$A = \begin{bmatrix} 0.9944 & -0.0067 & 0.0068 & 0.0052 \\ 0.0054 & 0.9991 & 0.0020 & 0.0021 \\ -0.0045 & 0.0028 & 0.9896 & 0.0098 \\ -0.0081 & 0.0005 & -0.0138 & 0.9672 \end{bmatrix}, \quad B = \begin{bmatrix} -0.1856 & 0.0765 & 0.1989 \\ 0.0709 & -0.0403 & 0.0318 \\ -0.1691 & 0.0319 & -0.0640 \\ -0.1559 & -0.0703 & 0.1537 \end{bmatrix}, \quad (20)$$

$$C = \begin{bmatrix} 0.2325 & 0.0730 & -0.1673 & -0.0252 \\ 0.1456 & 0.0427 & 0.0151 & -0.2131 \\ 0.0680 & 0.0200 & -0.0725 & 0.0851 \end{bmatrix}, \quad D = 0_{3,3}.$$

This system is obtained by discretizing the continuous-time system in [12] with the sampling period 0.005 [sec]. In particular, the realization (20) is already balanced, and the corresponding balanced Gramian Σ is

$$\Sigma = \begin{bmatrix} 7.2342 & 0 & 0 & 0 \\ 0 & 4.3586 & 0 & 0 \\ 0 & 0 & 1.6442 & 0 \\ 0 & 0 & 0 & 0.8323 \end{bmatrix}. \quad (21)$$

It can be seen that the smallest Hankel singular value is 0.8323 and its multiplicity is one.

With this in mind, we now construct the third-order system $G_3(z)$ that minimizes $\|G - G_3\|_\infty$, by means of the suggested two-step procedure in Theorem 2. Firstly, by solving the LMI (15) with $P_{12} = [I_3 \ 0_{3,1}]^T$, we obtained P_{11} and Q_{22} as follows[†]:

$$P_{11} = \begin{bmatrix} 7.2342 & 0 & 0 & 0 \\ 0 & 4.3586 & 0 & 0 \\ 0 & 0 & 1.6442 & 0 \\ 0 & 0 & 0 & 0.8323 \end{bmatrix}, \quad Q_{22} = \begin{bmatrix} 7.1384 & 0 & 0 \\ 0 & 4.1997 & 0 \\ 0 & 0 & 1.2229 \end{bmatrix}. \quad (22)$$

It should be noted that, from the proof of Lemma 3, the matrix P_{11} is expected to be very close to Σ , and this is surely achieved in (22). It is also observed that Q_{22} is very close to the inverse of P_{22} given in (14) as expected.

[†]All LMI-related computations are carried out with MATLAB Robust Control Toolbox R2006a.

With these P_{11} , Q_{22} and P_{12} , we next constructed \tilde{P} as in Theorem 2 and solved (6) for unknown A_3 , B_3 , C_3 and D_3 . Thus we have

$$\begin{aligned} A_3 &= \begin{bmatrix} 0.9938 & -0.0112 & 0.0351 \\ 0.0031 & 0.9991 & 0.0071 \\ -0.0010 & 0.0012 & 0.9822 \end{bmatrix}, & B_3 &= \begin{bmatrix} 1.4165 & -0.5217 & -1.5081 \\ -0.2863 & 0.1853 & -0.1601 \\ 0.2865 & -0.0475 & 0.0941 \end{bmatrix}, \\ C_3 &= \begin{bmatrix} -0.0330 & -0.0173 & 0.1333 \\ -0.0238 & -0.0095 & -0.0349 \\ -0.0082 & -0.0050 & 0.0679 \end{bmatrix}, & D_3 &= \begin{bmatrix} 0.0664 & 0.0240 & -0.0519 \\ 0.5001 & 0.2253 & -0.4925 \\ -0.1969 & -0.0909 & 0.1992 \end{bmatrix}. \end{aligned} \quad (23)$$

We can confirm that the third-order system $G_3(z)$ with these state-space matrices (A_3, B_3, C_3, D_3) surely satisfies $\|G - G_3\|_\infty = 0.8323$.

For comparison, we construct a third-order system $G_{3b}(z)$ via balanced truncation. It turns out that the associated error is given by $\|G - G_{3b}\|_\infty = 1.6281$, which is far from the optimal value 0.8323 achieved by $G_3(z)$. We next apply the alternating projection algorithm suggested in [12, 13]. To implement this algorithm, we first determine the initial matrices $(P_{11,0}, X_{11,0})$ of (7) by using the state space matrices of $G_{3b}(z)$. Namely, we construct an error system (4) using $G_{3b}(z)$ and solve the LMI (6). Then, by means of resulting P , we determine $(P_{11,0}, X_{11,0})$ through (7c), which turns out to be

$$\begin{aligned} P_{11,0} &= \begin{bmatrix} 7.3565 & -0.0203 & 0.0339 & -0.0758 \\ -0.0203 & 4.5347 & -0.0750 & 0.0636 \\ 0.0339 & -0.0750 & 2.2538 & -0.1172 \\ -0.0758 & 0.0636 & -0.1172 & 1.7135 \end{bmatrix}, \\ X_{11,0} &= \begin{bmatrix} 2.8315 & -0.0128 & -0.0044 & 0.0129 \\ -0.0128 & 1.7418 & -0.0028 & -0.0319 \\ -0.0044 & -0.0028 & 0.7558 & 0.0161 \\ 0.0129 & -0.0319 & 0.0161 & 0.5873 \end{bmatrix}. \end{aligned} \quad (24)$$

Using these matrices as initial values, we applied the alternating projection algorithm. It follows that the algorithm behaves well, and we successfully obtained matrices P_{11} and X_{11} that satisfy (7) for $\gamma = 0.8324$. Thus, using the alternating projection algorithm, we can also construct the optimal reduced-order system in this case. However, to the best of the authors' knowledge, definite proof for this assertion in general cases is not shown in the literature.

5 Conclusion

In this paper, we dealt with the \mathcal{H}_∞ model reduction problem for discrete-time LTI systems. By means of the recently developed LMI results for model reduction, we first showed a concise proof for the well-known lower bounds of the approximation error. We further

demonstrated that, in the case where we reduce the system order by the multiplicity of the smallest Hankel singular value, the \mathcal{H}_∞ optimal models can readily be constructed via LMI optimization. Even though these results could be deduced from [3], the contribution of the present paper should be significant and amounts to developing LMI techniques for purely discrete-time systems and giving explicit LMI formulas for the optimal \mathcal{H}_∞ model reduction. To explore possible extension of the present approach to model reduction of time-varying systems [5, 18, 21] and time-delay systems [6] is an interesting future topic.

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Appendix

Proof of Lemma 3. Let us define $\varepsilon := \gamma - \sigma_m > 0$ and consider the following matrix inequalities that correspond to (7) in Lemma 1:

$$-(P_{11} - P_{12}P_{22}^{-1}P_{12}^T)^{-1} + A(P_{11} - P_{12}P_{22}^{-1}P_{12}^T)^{-1}A^T + \frac{1}{(\sigma_m + \varepsilon)^2}BB^T < 0, \quad (25a)$$

$$-P_{11} + A^TP_{11}A + C^TC < 0, \quad (25b)$$

$$P_{11} - P_{12}P_{22}^{-1}P_{12}^T > 0. \quad (25c)$$

Then, to prove Lemma 3, it suffices to show that for any $\varepsilon > 0$, there exists $P_{11} \in \mathcal{S}_n$ satisfying (25) with P_{12} and P_{22} given in (14). To this end, let us first consider $\Pi > 0$ that satisfies the following Riccati equation and inequality constraint, which does exist if $Q > 0$ is small enough:

$$-\Pi + A^T\Pi A + A^T\Pi B(2\sigma_m I - B^T\Pi B)^{-1}B^T\Pi A + Q = 0, \quad 2\sigma_m I - B^T\Pi B > 0. \quad (26)$$

Then, it is not hard to see that $P_{11} := \Sigma + \varepsilon\Pi$ satisfies (25b), since we have from (12b) and (26) that

$$\begin{aligned} & -(\Sigma + \varepsilon\Pi) + A^T(\Sigma + \varepsilon\Pi)A + CC^T \\ & = -\varepsilon(A^T\Pi B(2\sigma_m I - B^T\Pi B)^{-1}B^T\Pi A + Q) < 0. \end{aligned} \quad (27)$$

The condition (25c) is also satisfied since (13) indicates that

$$\Sigma + \varepsilon\Pi - P_{12}P_{22}^{-1}P_{12}^T = \varepsilon\Pi + \sigma_m^2\Sigma^{-1} > 0. \quad (28)$$

To prove that (25a) holds, we first note that the following condition holds for any $\varepsilon > 0$:

$$\begin{bmatrix} \sigma_m^2\Sigma^{-1} & 0 & A^T \\ 0 & (\sigma_m^2 + \varepsilon^2)I & B^T \\ A & B & \sigma_m^{-2}\Sigma \end{bmatrix} \geq 0. \quad (29)$$

This can be easily verified since we have

$$\begin{bmatrix} \sigma_m^2 \Sigma^{-1} & 0 & A^T \\ 0 & (\sigma_m^2 + \varepsilon^2)I & B^T \\ A & B & \sigma_m^{-2} \Sigma \end{bmatrix} = V^T \begin{bmatrix} \sigma_m^2 \Sigma^{-1} & 0 & 0 \\ 0 & (\sigma_m^2 + \varepsilon^2)I & 0 \\ 0 & 0 & \frac{\varepsilon^2}{\sigma_m^2(\sigma_m^2 + \varepsilon^2)} BB^T \end{bmatrix} V \quad (30)$$

where

$$V = \begin{bmatrix} I & 0 & \sigma_m^{-2} \Sigma A^T \\ 0 & I & (\sigma_m^2 + \varepsilon^2)^{-1} B^T \\ 0 & 0 & I \end{bmatrix}. \quad (31)$$

Hence, we obtain from (29) that

$$\begin{aligned} & \begin{bmatrix} I & 0 \\ 0 & I \\ -\sigma_m^2 \Sigma^{-1} A & -\sigma_m^2 \Sigma^{-1} B \end{bmatrix}^T \begin{bmatrix} \sigma_m^2 \Sigma^{-1} & 0 & A^T \\ 0 & (\sigma_m^2 + \varepsilon^2)I & B^T \\ A & B & \sigma_m^{-2} \Sigma \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \\ -\sigma_m^2 \Sigma^{-1} A & -\sigma_m^2 \Sigma^{-1} B \end{bmatrix} \\ &= \begin{bmatrix} \sigma_m^2 \Sigma^{-1} - \sigma_m^2 A^T \Sigma^{-1} A & -\sigma_m^2 A^T \Sigma^{-1} B \\ -\sigma_m^2 B^T \Sigma^{-1} A & -\sigma_m^2 B^T \Sigma^{-1} B + (\sigma_m^2 + \varepsilon^2)I \end{bmatrix} \geq 0. \end{aligned} \quad (32)$$

On the other hand, it is apparent from (26) that the following inequality holds:

$$\begin{bmatrix} \Pi - A^T \Pi A & -A^T \Pi B \\ -B^T \Pi A & 2\sigma_m I - B^T \Pi B \end{bmatrix} > 0. \quad (33)$$

Thus, for any $\varepsilon > 0$, it follows from (32) and (33) that

$$\begin{aligned} & \begin{bmatrix} \sigma_m^2 \Sigma^{-1} - \sigma_m^2 A^T \Sigma^{-1} A & -\sigma_m^2 A^T \Sigma^{-1} B \\ -\sigma_m^2 B^T \Sigma^{-1} A & -\sigma_m^2 B^T \Sigma^{-1} B + (\sigma_m^2 + \varepsilon^2)I \end{bmatrix} \\ & + \varepsilon \begin{bmatrix} \Pi - A^T \Pi A & -A^T \Pi B \\ -B^T \Pi A & 2\sigma_m I - B^T \Pi B \end{bmatrix} > 0 \end{aligned} \quad (34)$$

or equivalently,

$$\begin{bmatrix} \varepsilon \Pi + \sigma_m^2 \Sigma^{-1} - A^T (\varepsilon \Pi + \sigma_m^2 \Sigma^{-1}) A & -A^T (\varepsilon \Pi + \sigma_m^2 \Sigma^{-1}) B \\ -B^T (\varepsilon \Pi + \sigma_m^2 \Sigma^{-1}) A & -B^T (\varepsilon \Pi + \sigma_m^2 \Sigma^{-1}) B + (\sigma_m + \varepsilon)^2 I \end{bmatrix} > 0. \quad (35)$$

Applying the Schur complement arguments to the above inequality by noting that $\varepsilon \Pi + \sigma_m^2 \Sigma^{-1} = \Sigma + \varepsilon \Pi - P_{12} P_{22}^{-1} P_{12}^T > 0$, we are lead to

$$\begin{bmatrix} (\Sigma + \varepsilon \Pi - P_{12} P_{22}^{-1} P_{12}^T)^{-1} & A & B \\ A^T & \Sigma + \varepsilon \Pi - P_{12} P_{22}^{-1} P_{12}^T & 0 \\ B^T & 0 & (\sigma_m + \varepsilon)^2 I \end{bmatrix} > 0. \quad (36)$$

It is apparent that the above inequality implies

$$-(\Sigma + \varepsilon \Pi - P_{12} P_{22}^{-1} P_{12}^T)^{-1} + A(\Sigma + \varepsilon \Pi - P_{12} P_{22}^{-1} P_{12}^T)^{-1} A^T + \frac{1}{(\sigma_m + \varepsilon)^2} BB^T < 0. \quad (37)$$

This clearly shows that $P_{11} = \Sigma + \varepsilon \Pi > 0$ satisfies (25) with P_{12} and P_{22} given by (14). Thus the proof is completed. Q.E.D.

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